Algorithms and Data Structures

Graphs 4: Minimal Spanning Trees

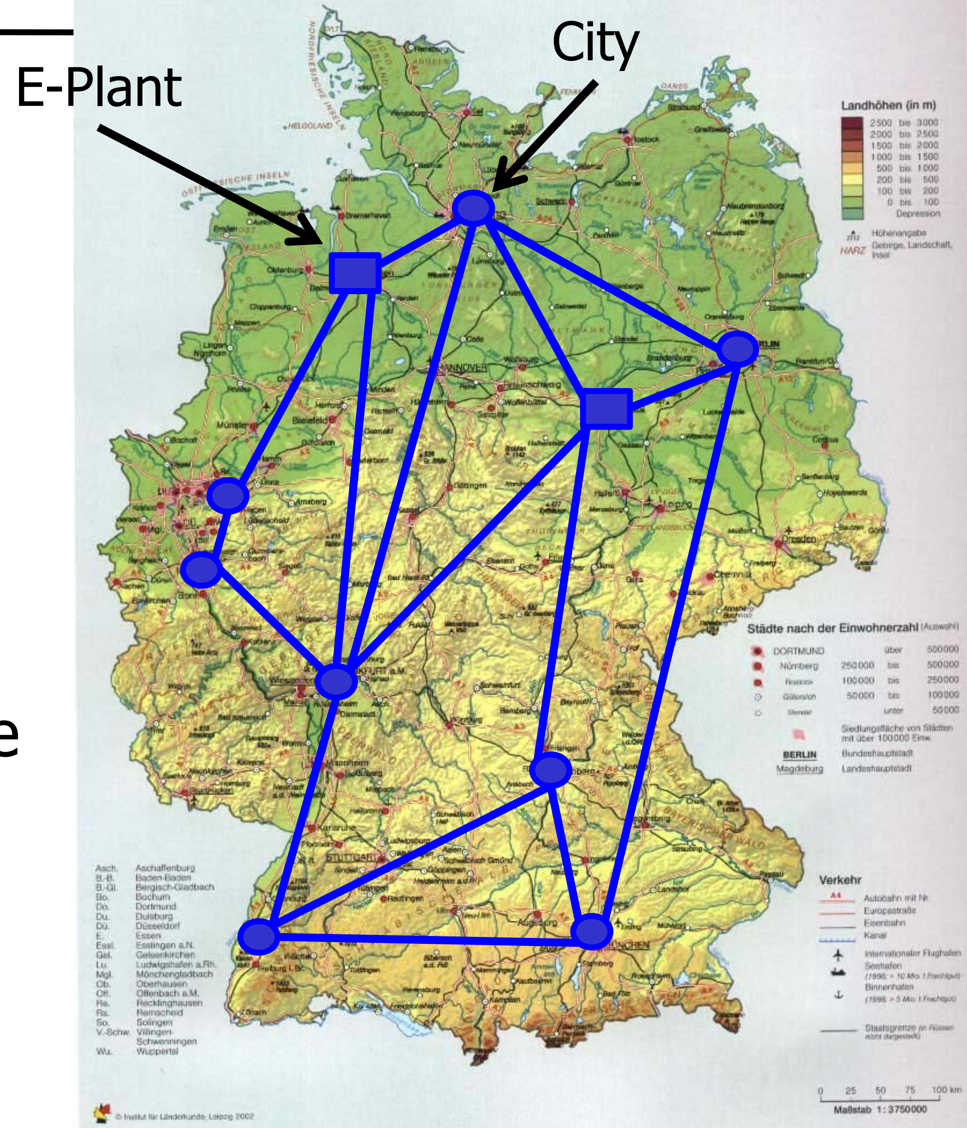
Marius Kloft

Die Energiewende

- Electricity is created in many more places than before
- Electricity is consumed in many places
- Places of production are not evenly distributed across the country
- Many say we need to build new electricity highways

Source: http://www.deutsche-mittelgebirge.de/

Die Energiewende



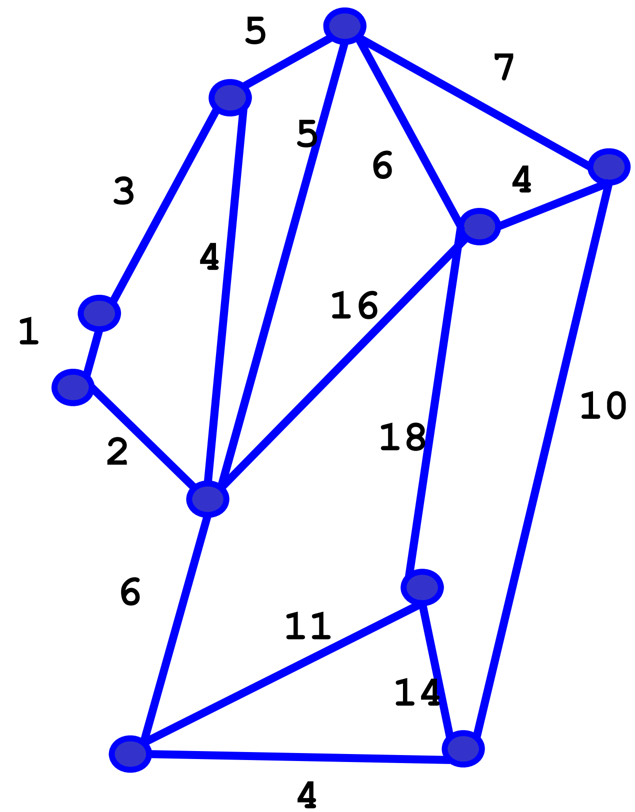
- How can we do this as cheap as possible?
- Not all connections are possible
  - Mountains, rivers, …
- Different connections have different costs

Die Energiewende

- Requirement for a solution: Every city and every plant must be connected to the network

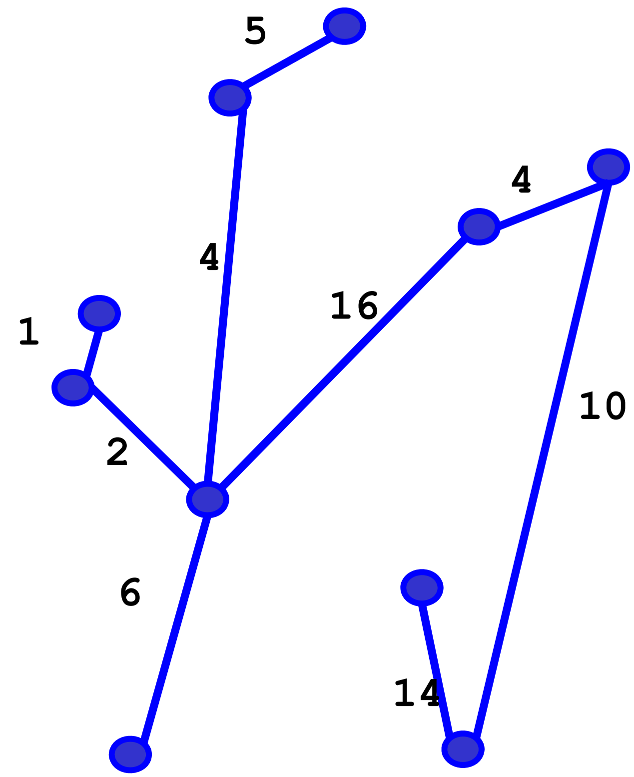
Abstraction

- Given an undirected, positively weighted, connected G=(V,E)
- Find a subset E'⊆E such that cost(E') is minimal and G'=(V, E') is connected
  - cost(E'): Sum of the edge weights
- E' (or G') is called a minimum spanning tree (MST) for G



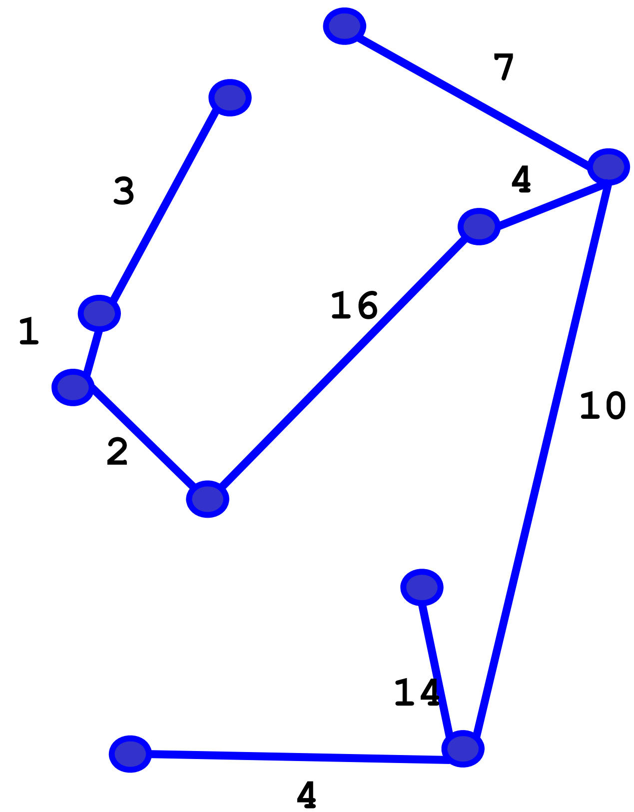
Example 1

- Cost = 62



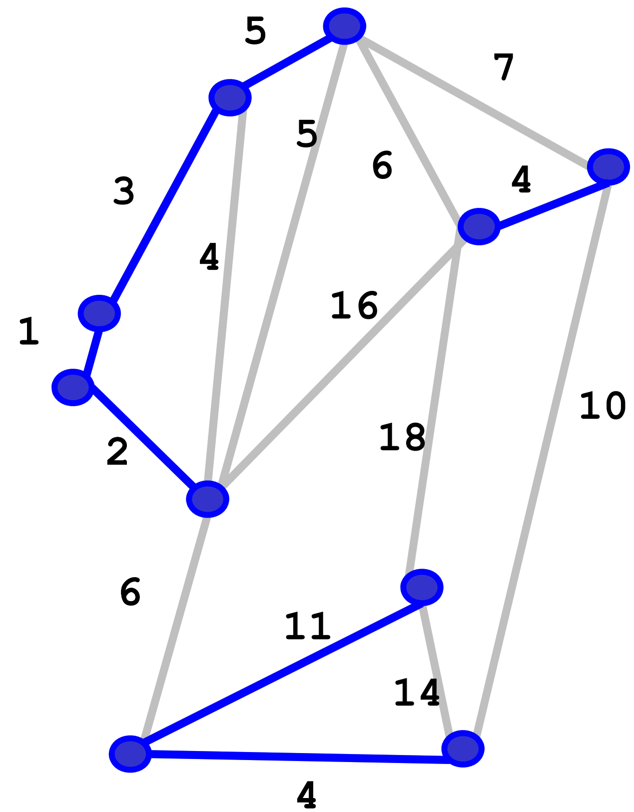
Example 2

- Cost = 61



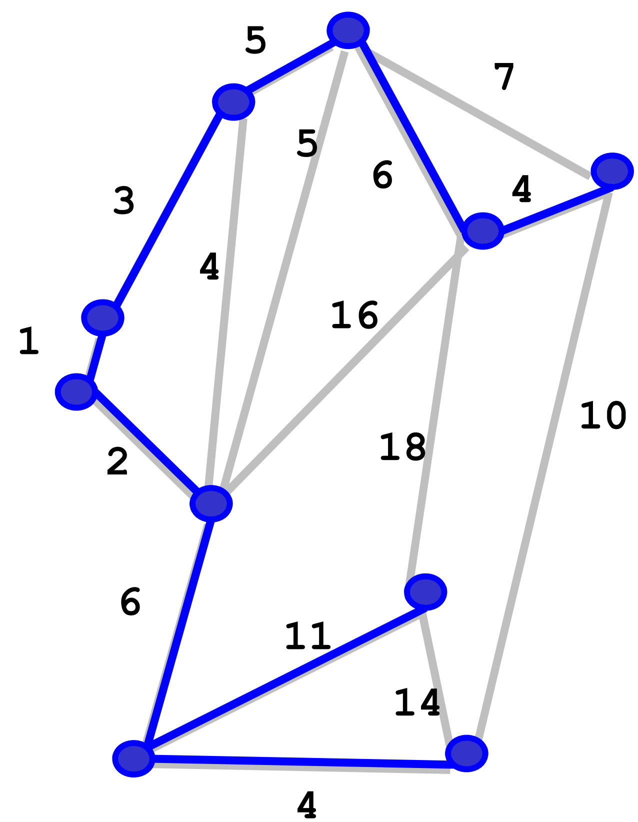
First Algorithm

- Let's try greedy
  - Sort edges by weight
  - Add edges to E' whenever it connects a new node to something
- Hmm



Second Algorithm

- Let‘s try greedy – another way
  - Sort edges by weight
  - Add cheapest edge to E’
  - Add edges to E’ in ascending order such that every new edge connects a new node with the graph induced by E’
  - Repeat until all nodes are connected
- Cost = 42
  - Is this optimal?
  - Does this always work?
  - How can we implement this algorithm efficiently?



Overview

- First algorithms for computing MST date back to the 1920s
- Algorithms are not very difficult; much research went into efficient implementations
- Actually, MSTs can be computed in a greedy manner
- Algorithms need not grow only one component; in general, we may have "connected islands" that all get connected to one component in the end
- In each step, one needs to decide which edge to add next to which island (or which edges not to add)
- What are criteria for adding / not adding edges?

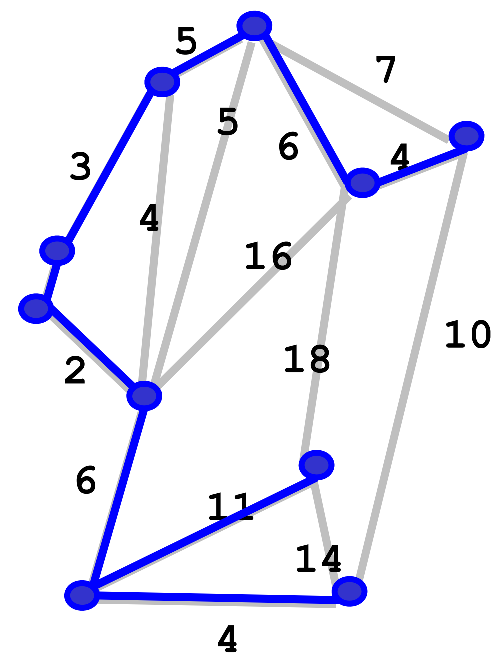
Content of this Lecture

- Minimal Spanning Trees
- Basic Properties
  - Tree
  - Cuts
  - Cycles
- Algorithms
- Implementation

Mimimal Spanning Tree

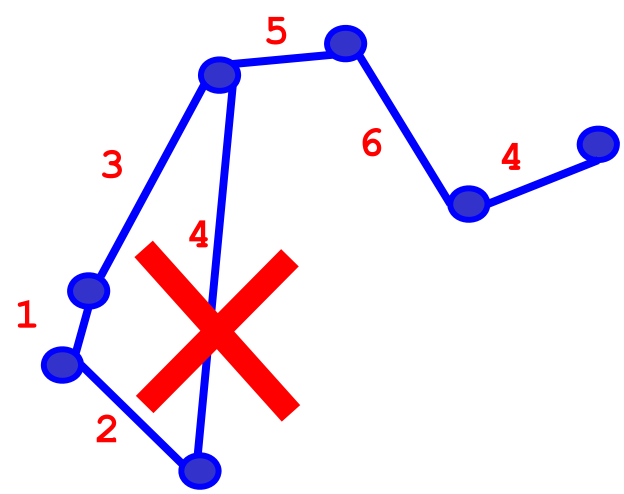
- Lemma
  Let G=(V, E) and let E′⊆E be the subset of E with minimal cost such that G′, the graph induced by E′, is connected. Then G′ is a tree (called "minimal spanning tree", MST).

- Proof
  - Recall: A (undirected) tree is a undirected, connected acyclic graph
  - By definition, G′ is connected and undirected
  - Need to show that G′ contains no cycle



Proof: MST is a Tree

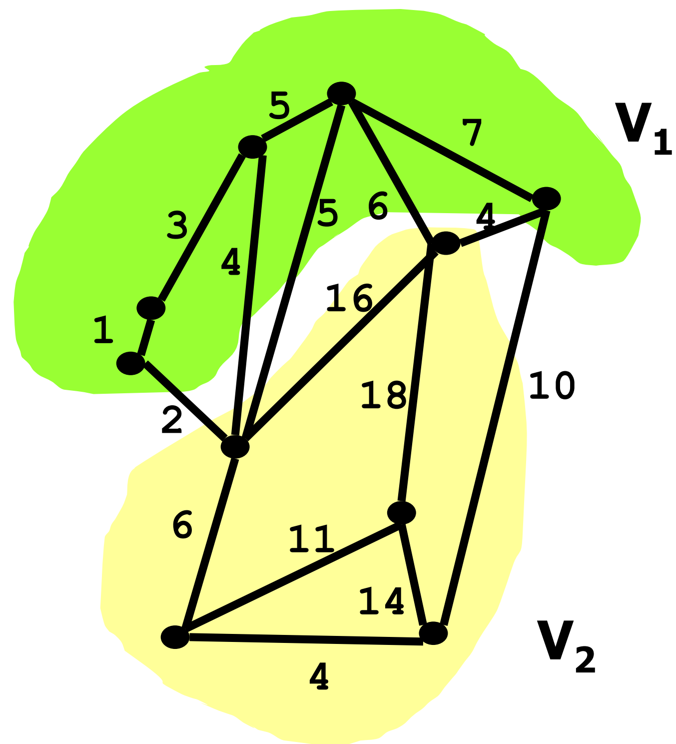
- Imagine G′ had a cycle. Then G′ cannot have minimal cost
  - because removing any of the edges of the cycle from E′ would create a subset E″ that has less cost (since we assumed all edge weights to be positive), and the induced subgraph would still be connected
- Contradiction



- Remark: If all edge weights are distinct, the MST is unique

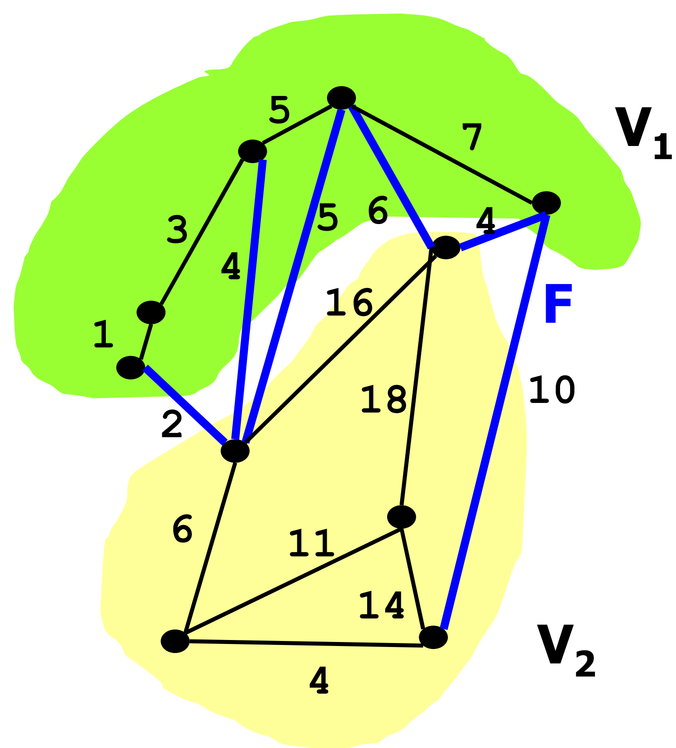
Cuts & Crossing Bridges

- Definition
  Let G=(V, E). A cut is a binary partition of V into sets V_1, V_2 such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$.



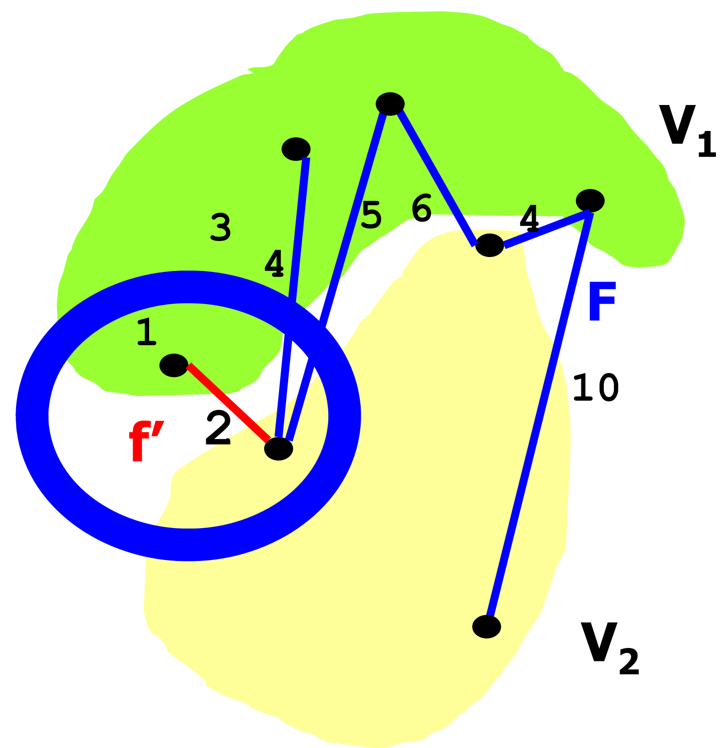
Cuts & Crossing Bridges

- Definition
  Let $G=(V, E)$. A cut is a binary partition of V into sets V_1, V_2 such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$.
- Definition
  Let $G=(V, E)$ and V_1, V_2 be a cut of V. Any edge connecting a node in V_1 to a node in V_2 is called crossing bridge. We denote the set of all crossing bridges by F.



Cut Property on Minimal Crossing Bridges

- Lemma (Cut Property)
  Let G=(V, E), let V_1, V_2 be a cut of V with crossing bridges F. Let F′ be those edges of F with minimal weight. Then:
  1) *Any MST G′ of G must contain at least one f′∈F′*
  2) *Every f′∈F′ is contained in at least one MST of G*
- Remarks
  - This holds for arbitrary cuts – a very powerful statement



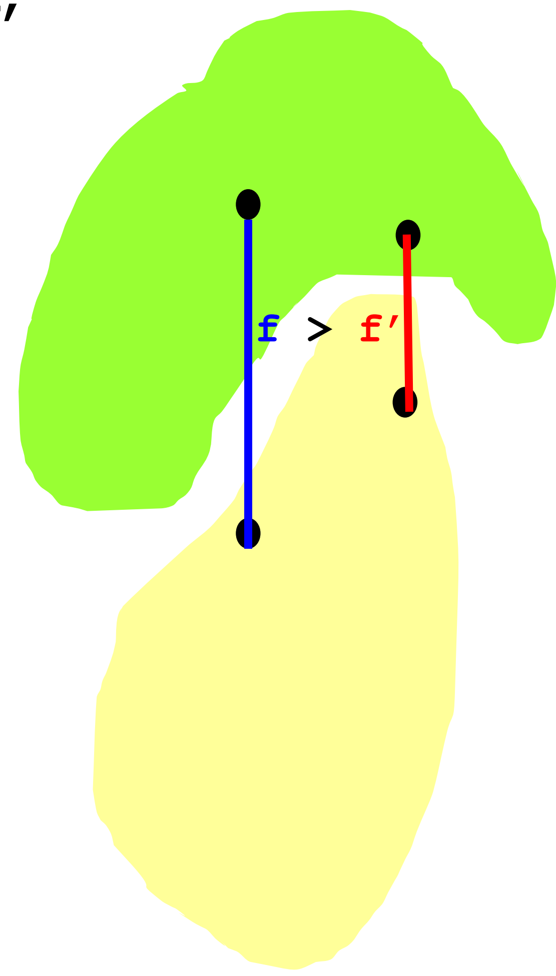
Proof, 1a)

1) Every MST G′ contains at least one f′∈F′

- Assume the contrary (G′ has no such f′) and let f′∈F′
- Still, G′ is connected, so it must contain at least one of the crossing edges from F

(a) Assume G′ contains only one f∈F

- f must have a higher weight than f′ because – by assumption - f∉F′
- Furthermore, because – by assumption – f is the only crossing edge, V_1 and V_2 must be connected in themselves
- Thus, removing f and adding some f′∈F′ creates a cheaper MST, so G′ cannot be minimal – contradiction.

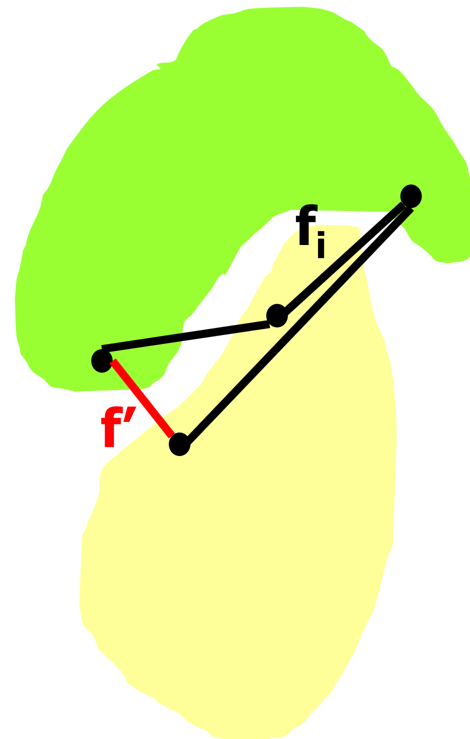


Proof, 1b)

1) Every MST G′ contains at least one f′∈F′

(b) The proof is similar if G′ contains multiple $f_i \in F$

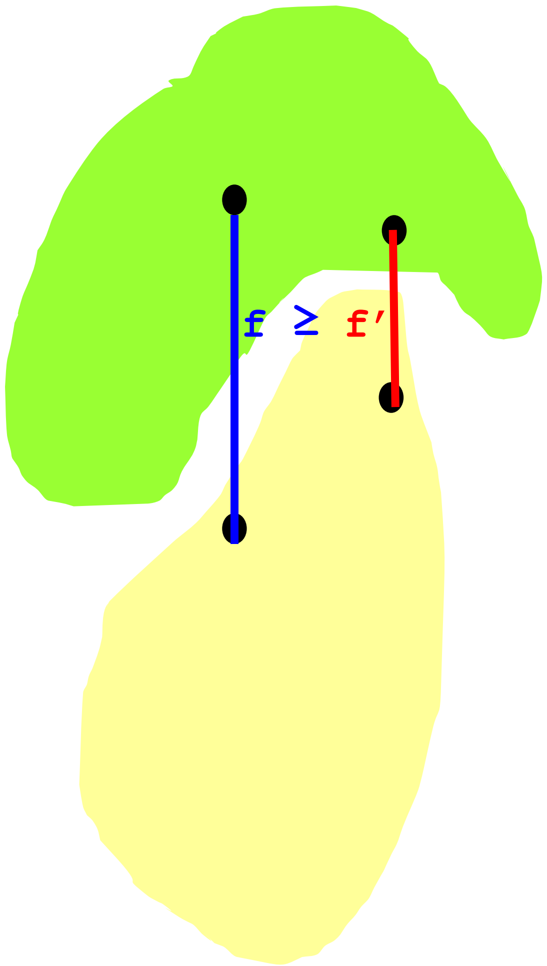
- Write f′=(v,v′)
- Since G′ is connected there exists a path p in G′ from v to v′
- Since f′ is a crossing bridge, v and v′ must lie on opposite sides of the cut
  - So the path p contains a crossing bridge $f_i \in F$
- Removing f_i from MST G′ breaks G′ into two components, and adding f′ re-connects them
  - resulting in cheaper MST (since f′ has smaller weight than f_i because $f_i \notin F'$)
  - Contradiction



Proof, 2)

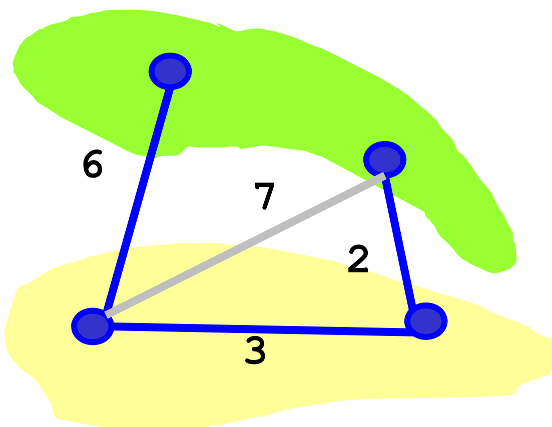
(2) Every f'∈F' is contained in at least one MST of G

- Imagine f' is not contained in any MST
- Let G' be such an MST
- Proof uses analogue argument as in (1):
  - Consider f∈F connecting V_1 and V_2
  - Removing f_i from G' breaks G' into two components, and adding f' re-connects them, resulting in G'' with equal or cheaper cost as G'
  - Thus G'' is an MST - Contradiction



Beware

- For a cut V_1, V_2, an MST G` may (have to) contain more than one crossing edge (but one must have minimal weight)



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- Minimal Spanning Trees
- Basic Properties
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  - Cuts
  - Cycles
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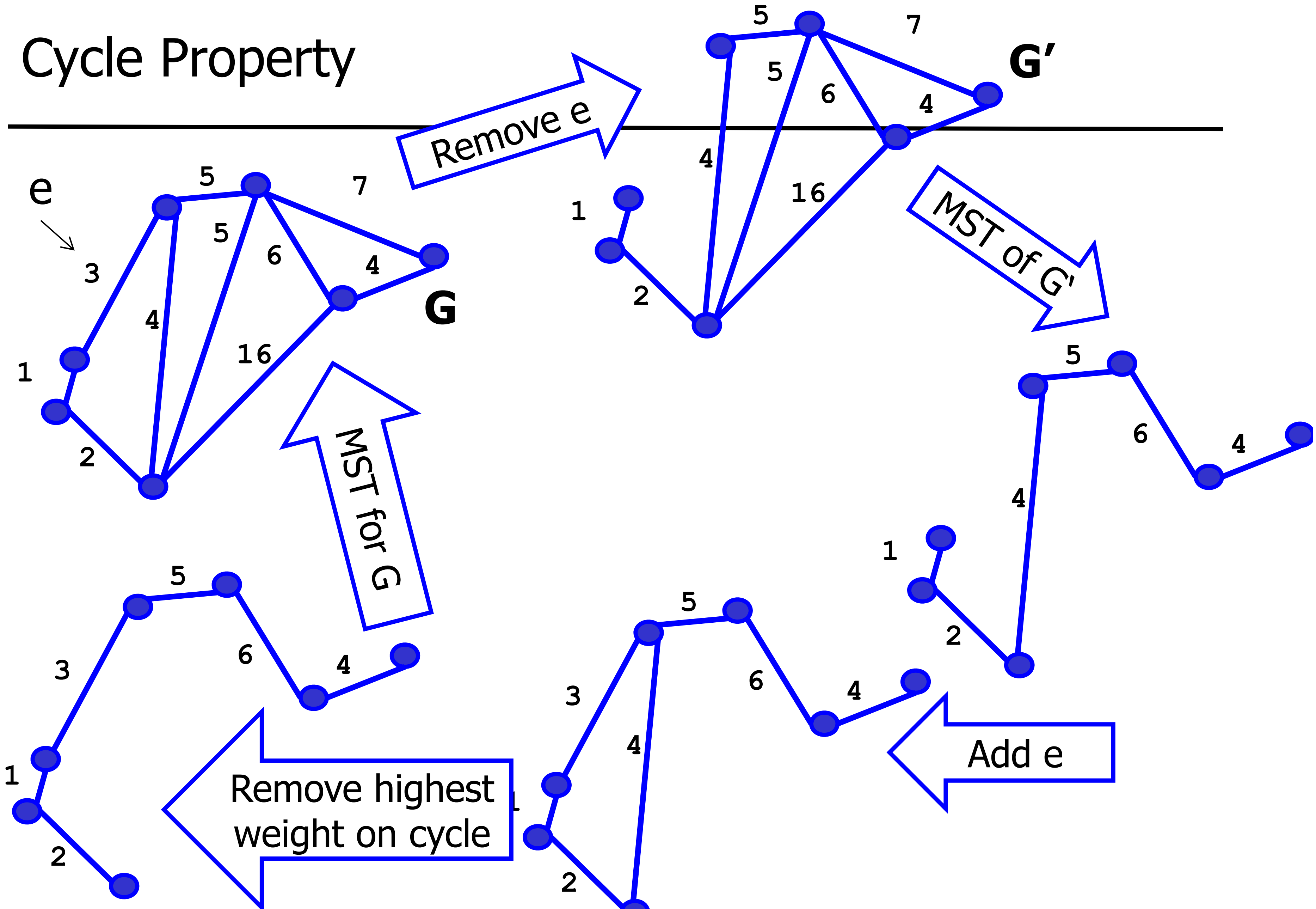
- Implementation

Cycles

- Lemma (cycle property)
  Let G=(V, E) and G'=(V, E') with E'=E|e for some edge e such that G' still is connected. Let T' be an MST for G'. When we add e to T' and remove the edge with the highest weight on the then introduced cycle in T', forming T, then T is an MST for G.
- Proof idea
  - Adding e must build a cycle because T' is MST over the same V
  - Removing any of the edges on the cycle still leaves a connected tree
  - Removing the most expensive one leaves the minimal tree

Cycle Property



Implications

- Note that T' is an MST for G without e
- Imagine we would enumerate edges by some order
- Taking into account a new e allows us to replace an edge in T' with a cheaper one, creating a "better" MST for G
  - If e is not the edge with the highest weight on the cycle
- This means that an edge with maximal weight on a cycle in G cannot be part of any MST of G

Content of this Lecture

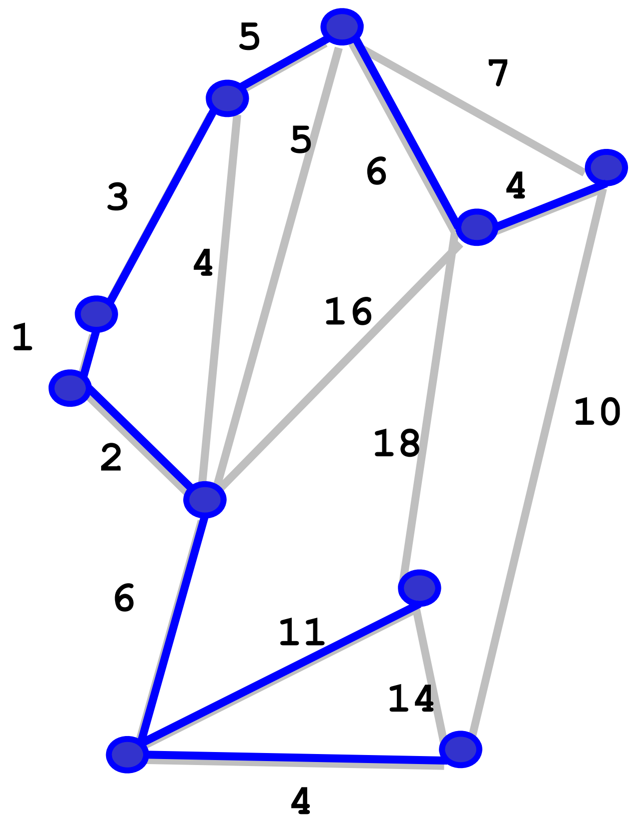
- Minimal Spanning Trees
- Basic Properties
- Algorithms
  - R.C. Prim: Shortest connection networks and some generalizations. Bell System Technical Journal, 1957
    - Also Jarnik, Prim, Dijkstra: Jarník, 1930 – Prim, 1957 – Dijkstra , 1959
  - J. Kruskal: On the shortest spanning subtree and the traveling salesman problem. Proc. of the American Mathematical Soc., 1956
  - Otakar Borůvka: O jistém problému minimálním (Über ein gewisses Minimierungsproblem), 1926
  - [Wikipedia, OW93, Sed04, Cor03]
- Implementation

Prim's Algorithm

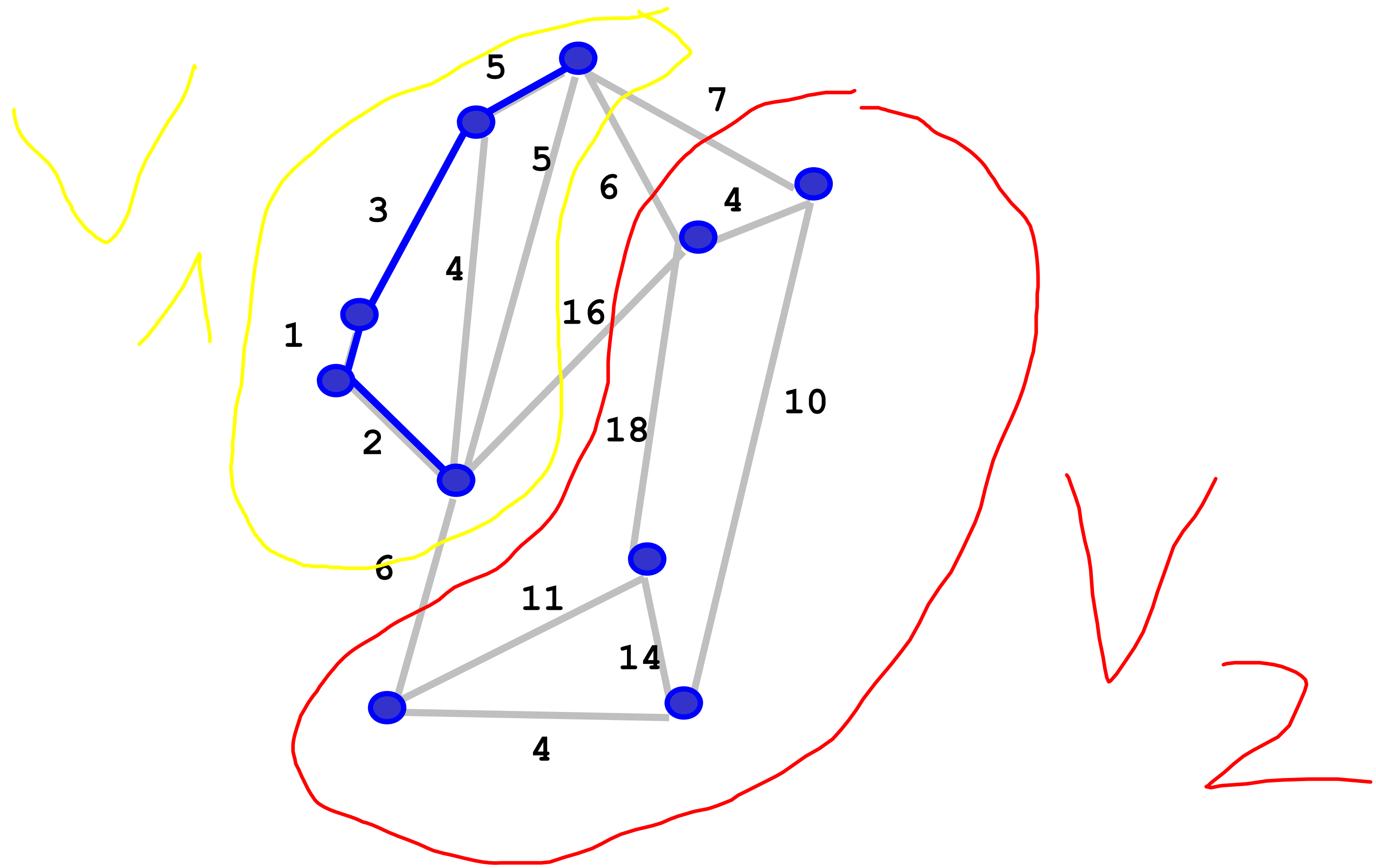
Greedy; we never make mistakes

- Prim's Algorithm
  Start with an empty tree T. Continue adding the edge e with the lowest cost to T such that e connects T with a new node until all nodes of G are in T. Then T is an MST
- Proof
  - Consider, at each stage, nodes in T as one partition V_1 and all other nodes as the other partition V_2
  - By cut-property lemma, the cheapest crossing-edge between V_1 and V_2 must be in an MST of G
  - Since we only add those edges, T finally must be an MST

Prim‘s Algorithm: Example

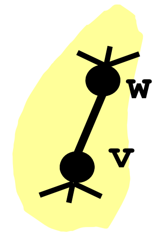


Prim‘s Algorithm: Example

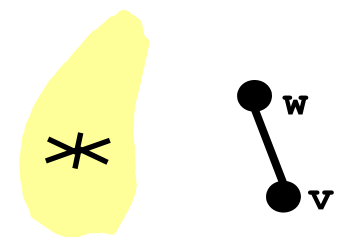


Kruskal's Algorithm

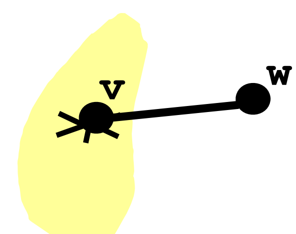
- *Start with an empty forest F. Continue "adding" edges e to F in order of increasing cost until F becomes a tree. Adding an edge e=(v, w) to F proceeds as follows:*
  - *Case 1: If F already contains a tree containing both v and w, then e is dropped*



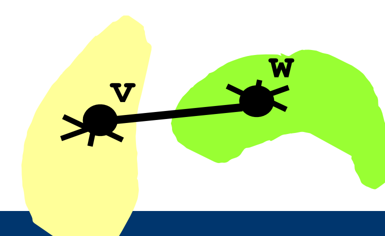
  - *Case 2: If no tree in F contains either v or w, then a new tree formed by e is added to F*



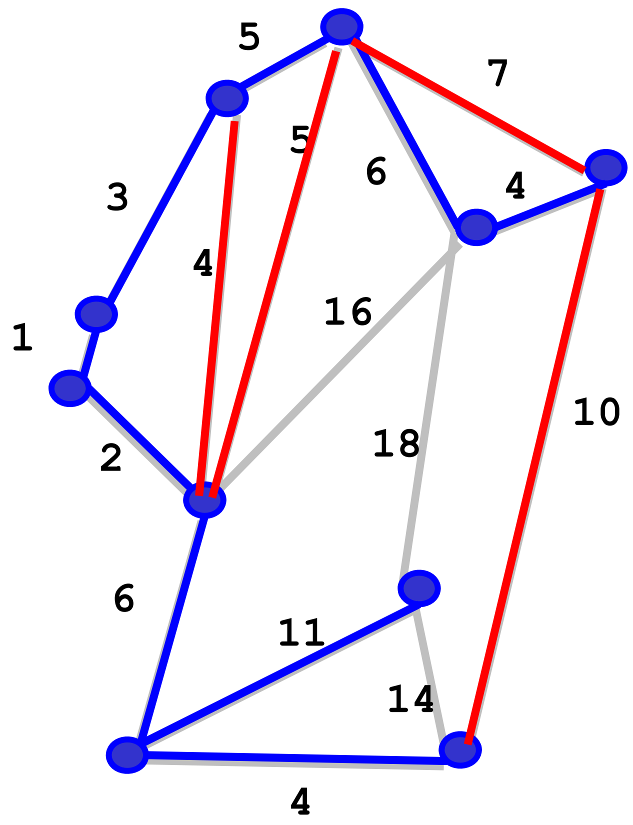
  - *Case 3: If F contains a tree T containing either v or w and neither T nor any other tree in F contains the other node, then e is added to T*



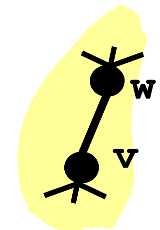
  - *Case 4: If F contains a tree T containing either v or w and a tree T' containing the other node, then T, T' and e are merged into one tree*



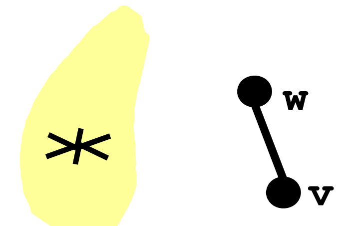
Kruskal's Algorithm: Example



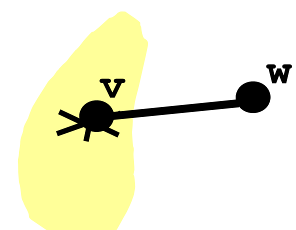
Proof by Induction (Only Central Idea)



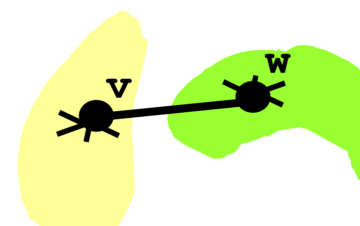
- We show that each of the trees in F is an MST of a subgraph of G
- Claim is true at the beginning (F empty)
- Assume claim holds before we consider next edge e=(v, w)



- Case 1: Claim holds, because e would introduce a cycle, and e has the highest cost on this cycle (all cheaper edges were considered before). Thus, e cannot be in an MST of G



- Case 2: Claim holds because e is the cheapest edge connecting v and w, and thus the new tree is an MST (for subgraph induced by {v,w})

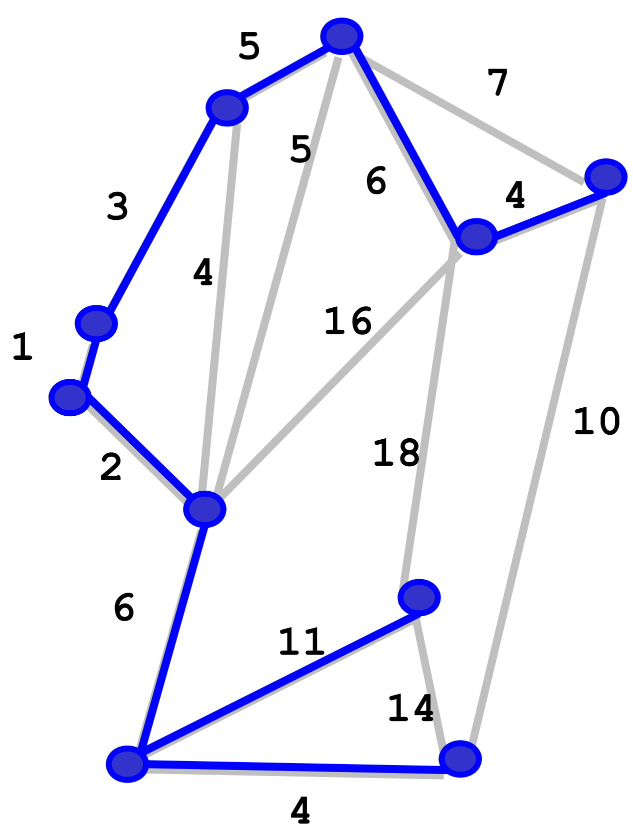


- Case 3: Claim holds because e is the cheapest edge con necting v (or w) and T, and thus the new tree is an MST
- Case 4: Claim holds because e is the cheapest edge connecting T and T', and thus the new tree is an MST

Boruvka‘s Algorithm

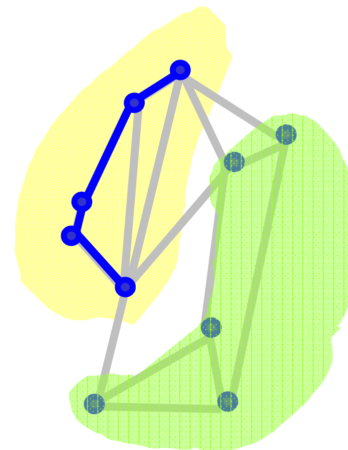
- Boruvka‘s Algorithm
  Start with an empty forest F. Add all edges (at once) that connect a node with its "cheapest" neighbor (edge with least cost) – taking care of not introducing cycles. Then consider each pair of trees in F and add cheapest crossing-edge until F becomes a unique tree.
- Proof (and details) omitted; see [Sed04]

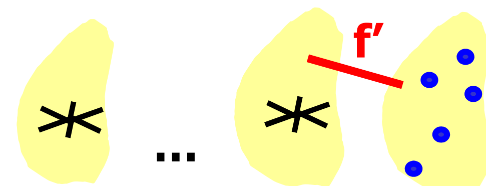
Boruvka‘s Algorithm: Example



Communalities

- All three algorithms iteratively choose an edge by the cut property or reject an edge by the cycle property
  - Prim: Growing T is one partition, all other nodes the other (isolated nodes)
  - Kruskal: Each T that grows is one partition, all other nodes the other (islands of mini-MSTs)
  - Boruvka: Each T that grows is one partition, all other nodes the other (islands of mini-MSTs)
- Difference is the order in which edges are chosen – there are always many candidates
- Differences are the data structures that these algorithms need to maintain





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- Minimal Spanning Trees
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  - Prim's, Kruskal's

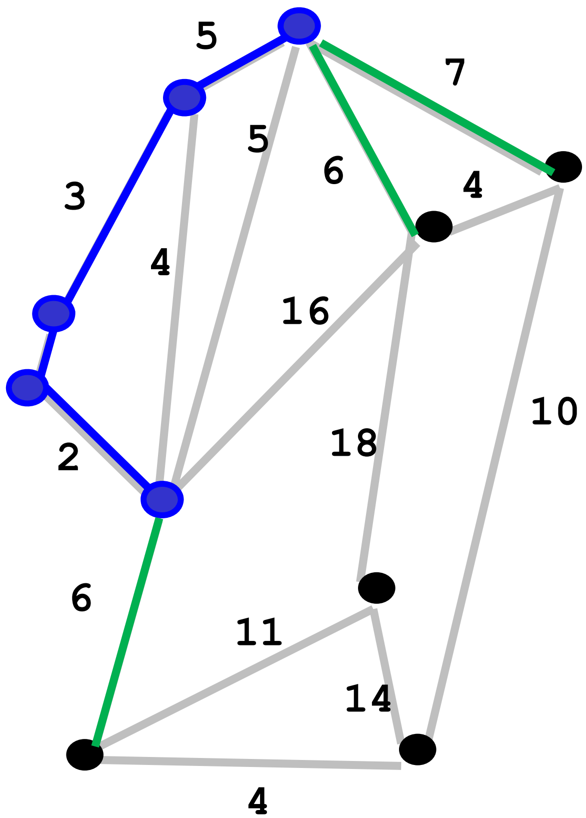
Implementing Prim`s Algorithm

- ChooseCheapest: Choose cheapest edge connecting a node in T to a node not yet in T
- Brute force: Search all such edges in every step
- More clever
  - Maintain a PQ of nodes reachable by one edge from T sorted by cost
  - When adding a new node to T, look at its neighbors and add them to the PQ (if not reachable before) or update costs (if now there is a cheaper edge reaching them)

```
G := (V, E);
T := ∅;
R := E;
for i = 1 to |V|-1 do
  e := chooseCheapest( T, R);
  T := T ∪ e;
  R := R \ e;
end for;
```

Example

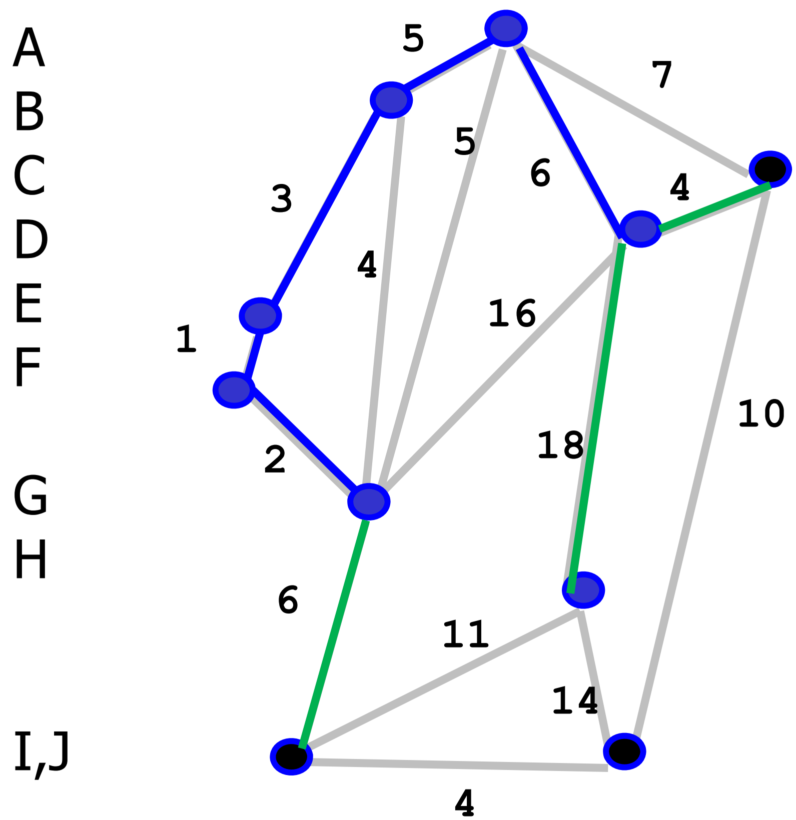
A
B
C
D
E
F
G
H
I,J



- T = {A, F, E, B, G}
- PQ = {(D,6), (I, 6), (C, 7)}

- Choose (A-D, 6)

Example



- T = {A, F, E, B, G}
- PQ = {(D,6), (I, 6), (C, 7)}

- Choose (A-D, 6)
- New T: {A, F, E, B, G, D}
- PQ = {(C,4), (I, 6), (H, 18)}

Complexity

- n=|V|, m=|E|
- Prim' algorithm runs in O((n+m)*log(n))
  - n times through the loop, performing altogether at most m PQ-operations in log(n)

Implementing Kruskal‘s Algorithm

- ChooseCheapest: Simply choose cheapest edge in E
  - I.e., sort E at the beginning
- UNION-FIND data structure
  - Maintains a set of sets (all trees T)
  - Needs a method for quickly finding the set containing a given element (find)
  - Needs a method for quickly merging two sets (union)
- Can be implemented in O(m*log(n))

```
G := (V, E);
for i = 1 to |V| do
  T[i] := {i};
end do:
repeat
  (v,w) := chooseCheapest( E);
  E := E \ (v,w);
  T := find( v);
  T′ := find (w);
  if T≠T′ then
    T := T ∪ T′;
  end if;
until |T|=|V|;
```